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SU(N) confining strings

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Abstract: String representation of the SU(N)-version of compact QED in three dimensions is constructed. It is demonstrated how the resulting SU(N)-generalization of the so-called theory of confining strings can be obtained in various ways. In its weak-field limit (corresponding to the limit of low monopole densities), this theory enables one to fix the value of the string tension, which cannot be fixed when being deduced from the mean magnetic field inside a flat contour. Moreover, the obtained theory enables one to find also the coupling constants of the terms in the expansion of the nonlocal string effective action which are higher in the derivatives than the Nambu-Goto term (some of which vanish at the flat surface).

Keywords: Confinement, Nonperturbative Effects, Field Theories in Lower Dimensions, Solitons Monopoles and Instantons.

Contents

1. Introduction

One of the most powerful methods, aimed at the description of confinement in QCD, is the method of abelian projections [1] (see ref. [2] for the latest developments and ref. [3] for reviews). Among various ways to model the grand canonical ensemble of abelian-projected monopoles, the most natural one is to treat them as the Coulomb plasma, analogous to the one of compact QED [4] (see ref. [5] for a review). The string description of the latter in terms of the Kalb-Ramond field, the so-called theory of confining strings, has been proposed for the 3D-case in ref. [6] and then generalized to the 4D-case in ref. [7]. It should however been mentioned that the ideas to describe confinement in terms of the Kalb-Ramond field were put forward almost 30 years ago [8]. These ideas have been elaborated further on in ref. [9], where the connection between magnetic condensation and the Kalb-Ramond formulation of confinement has for the first time been pointed out. Moreover, in ref. [9] a non-abelian generalization of the Kalb-Ramond field has been proposed and used for the construction of a string representation of the Wilson loop on the lattice. It should be noticed that such a non-abelian formulation of confining strings is different from its abelian-type counterpart, which for the SU(2)-case has been suggested in ref. [6].

A problem remained since the time of the paper [6] is the generalization of the (abelian-type) theory of confining strings from the SU(2)-case to an arbitrary number of colors. Some progress in this direction has been achieved in refs. [10, 11]. In particular, in ref. [10], it has been shown that at large-N the high-temperature behavior of the free energy of confining strings matches that of QCD. A related problem is the investigation of the string representation of the $(2{+}1)D$ Georgi-Glashow model. This model is different from compact QED due to the presence of the charged W-bosons and due to the Higgs-mediated interaction of monopoles which becomes relevant when the Higgs boson is not infinitely heavy. In this way, the influence of W-bosons to the deconfining phase transition has been analysed in the confining-string language in ref. [12]. The string representation of the BPS-limit [13] (where the mass of the Higgs boson is much smaller than the mass of the W-boson) of the SU(3)-version of the $(2{+}1)D$ Georgi-Glashow model has been found in [14]. In ref. [15], the representation of the SU(N)-version of this model in terms of the monopole densities has been derived, and the potential of these densities has been discussed.

The aim of the present paper is to derive and investigate the theory of confining strings which would describe the string content of the SU(N)-generalization of $3D$ compact QED. The paper is organized as follows. In section 2, we shall find the string tension of the Wilson loop defined at a flat contour (henceforce called for shortness "flat Wilson loop"). This will be done by the method based on the solution of the system of the saddle-point equations for the dual-photon and the magnetic field. (When the magnetic field is eliminated from these equations, the resulting equation for the dual-photon field is analogous to the one derived for the usual compact QED in ref. [4].) In section 3, we shall develop the theory of confining strings based on the Kalb-Ramond field. This will be done by two various methods, both of which illustrate how the Kalb-Ramond field unifies the monopole and the free-photonic contributions to the Wilson loop. The string tension derivable in the weak-field limit of this theory (which is nothing else, but the limit of low monopole densities) is parametrically the same as the one obtained in section 2 for the flat loop. However, the advantage of the formalism based on the Kalb-Ramond field is that it enables one not only to fix the proportionality coefficient at the string tension, but also to derive the coupling constants of the higher-derivative string terms.

In summary, the main results of the paper will be discussed.

2. String tension of the flat Wilson loop

Since the string tension is generated by monopoles, in this section we shall omit photons. The partition function of the grand canonical ensemble of monopoles in the Euclidean space-time reads

$$\mathcal{Z}_{\rm mon} = \sum_{\mathcal{N}=0}^{\infty} \frac{\zeta^{\mathcal{N}}}{\mathcal{N}!} \left\langle \exp\left[-\frac{g_m^2}{2}\int d^3x d^3y \vec{\rho}^{\,\mathcal{N}}(\vec{x}) D_0(\vec{x}-\vec{y}) \vec{\rho}^{\,\mathcal{N}}(\vec{y})\right]\right\rangle_{\rm mon}, \tag{2.1}$$

where the monopole density is defined as $\vec{\rho}^{\,\mathcal{N}}(\vec{x}) = \sum_{k=1}^{\mathcal{N}} \vec{q}_{i_k} \delta(\vec{x}-\vec{z}_k)$ at $\mathcal{N} \geq 1$ and $\vec{\rho}^{\,\mathcal{N}=0}(\vec{x}) = 0$. In these equations, g_m is the magnetic coupling constant of dimensionality $[\text{mass}]^{-1/2}$, $gg_m = 4\pi$, $\zeta \propto \mathrm{e}^{-\text{const}/g^2}$ is the monopole fugacity of dimensionality $[\text{mass}]^3$, $D_0(\vec{x}) = 1/(4\pi|\vec{x}|)$ is the $3D$ Coulomb propagator, and it has been taken into account that the monopole charges are distributed along the $(N-1)$-dimensional root vectors $\vec{q}_i$'s of the group SU(N) [17]. Furthermore, the average is defined as follows:

$$\langle \mathcal{O} \rangle_{\rm mon} = \prod_{n=0}^{\mathcal{N}} \int d^3 z_n \sum_{i_n = \pm 1, \ldots, \pm \frac{N(N-1)}{2}} \mathcal{O}\,. \tag{2.2}$$

Upon the explicit summation, the partition function (2.1) can be represented in the Sine-Gordon–type form

$$\mathcal{Z}_{\rm mon} = \int \mathcal{D}\vec{\chi} \exp\left[-\int d^3x \left(\frac{1}{2}(\partial_\mu \vec{\chi})^2 - 2\zeta \sum_{i=1}^{N(N-1)/2} \cos(g_m \vec{q}_i \vec{\chi})\right)\right], \tag{2.3}$$

where $\vec{\chi}$ is the dual-photon field. By virtue of the formula [18] $\sum_{i=1}^{N(N-1)/2} q_i^\alpha q_i^\beta = \frac{N}{2}\delta^{\alpha\beta}$, the following value of the Debye mass of the dual photon has been obtained [15]: $m_D = g_m\sqrt{N\zeta}$.

One can further introduce the monopole field-strength tensor $\vec{F}_{\mu\nu}^{\mathcal{N}}$ violating the Bianchi identity as $\frac{1}{2}\varepsilon_{\mu\nu\lambda}\partial_\mu \vec{F}_{\nu\lambda}^{\mathcal{N}} = g_m\vec{\rho}^{\mathcal{N}}$. Since it was argued to omit photons throughout this section, we obtain for $\vec{F}_{\mu\nu}^{\mathcal{N}}$ the following expression:

$$\vec{F}_{\mu\nu}^{\mathcal{N}}(\vec{x}) = -g_m\varepsilon_{\mu\nu\lambda}\partial_\lambda \int d^3y D_0(\vec{x}-\vec{y})\,\vec{\rho}^{\mathcal{N}}(\vec{y}). \tag{2.4}$$

Then, by virtue of the Stokes' theorem and the formula

$$\mathrm{tr}\,\exp\left(i\vec{a}\vec{H}\right) = \sum_{\alpha=1}^{N} \exp\left(i\vec{a}\vec{\mu}_\alpha\right),$$

where $\vec{a}$ is an arbitrary $(N-1)$-component vector, $\vec{H}$ is the vector of diagonal generators and $\vec{\mu}_\alpha$'s are the weight vectors of the group SU(N), $\alpha = 1,\ldots,N$, we obtain the following expression for the Wilson loop defined at the $\mathcal{N}$-monopole configuration:

$$W(C)_{\mathrm{mon}}^{\mathcal{N}} = \frac{1}{N}\,\mathrm{tr}\,\exp\left(\frac{ig}{2}\vec{H}\int d^3x \vec{F}_{\mu\nu}^{\mathcal{N}}\Sigma_{\mu\nu}\right) = \frac{1}{N}\sum_{\alpha=1}^{N} W_\alpha^{\mathcal{N}}. \tag{2.5}$$

Here,

$$W_\alpha^{\mathcal{N}} \equiv \exp\left(\frac{ig}{2}\vec{\mu}_\alpha\int d^3x \vec{F}_{\mu\nu}^{\mathcal{N}}\Sigma_{\mu\nu}\right) \tag{2.6}$$

and

$$\Sigma_{\mu\nu}(\vec{x}) = \int_{\Sigma(C)} d\sigma_{\mu\nu}(\vec{x}(\xi))\,\delta(\vec{x}-\vec{x}(\xi))$$

is the vorticity tensor current defined at a certain surface $\Sigma(C)$ bounded by the contour C and parametrized by the vector $\vec{x}(\xi)$ with $\xi = \left(\xi^1,\xi^2\right)$ standing for the 2D coordinate. A straightforward calculation with the use of eq. (2.4) yields

$$W_\alpha^{\mathcal{N}} = \exp\left(i\vec{\mu}_\alpha\int d^3x\vec{\rho}^{\mathcal{N}}\eta\right), \tag{2.7}$$

where

$$\eta(\vec{x};C) = \int_{\Sigma(C)} d\sigma_\mu(\vec{x}(\xi))\,\partial_\mu\frac{1}{|\vec{x}-\vec{x}(\xi)|}$$

is the solid angle under which the surface $\Sigma(C)$ is seen by an observer located at the point $\vec{x}$, $d\sigma_\mu = \frac{1}{2}\varepsilon_{\mu\nu\lambda}d\sigma_{\nu\lambda}$. (Note that owing to the Gauss' law, $\eta \equiv 4\pi$ for a closed surface, i.e., when C is shrunk to a point.) The ratio of two $W_\alpha^{\mathcal{N}}$'s defined at different surfaces, Σ_1 and Σ_2, bounded by the contour C, reads

$$\begin{aligned}\frac{W_\alpha^{\mathcal{N}}(\Sigma_1)}{W_\alpha^{\mathcal{N}}(\Sigma_2)} &= \exp\left(i\vec{\mu}_\alpha\int_{\Sigma_1\cup\Sigma_2} d\sigma_\mu(\vec{x}(\xi))\int d^3x\vec{\rho}^{\mathcal{N}}(\vec{x})\,\partial_\mu\frac{1}{|\vec{x}-\vec{x}(\xi)|}\right)\\ &= \prod_{k=1}^{\mathcal{N}}\exp\left(-i\vec{\mu}_\alpha\vec{q}_{i_k}\int_{\Sigma_1\cup\Sigma_2} d\sigma_\mu(\vec{x}(\xi))\,\partial_\mu^{\vec{x}(\xi)}\frac{1}{|\vec{z}_k-\vec{x}(\xi)|}\right).\end{aligned}$$

Due to the Gauss' law, the last integral in this equation is equal either to -4π or to 0, depending on whether the point $\vec{z}_k$ lies inside or outside the volume bounded by the surface $\Sigma_1 \cup \Sigma_2$. Since the product $\vec{\mu}_\alpha \vec{q}_{i_k}$ is equal either to $\pm\frac{1}{2}$ or to 0, we conclude that $\frac{W_\alpha^{\mathcal{N}}(\Sigma_1)}{W_\alpha^{\mathcal{N}}(\Sigma_2)} = 1$. This fact proves the independence of $W_\alpha^{\mathcal{N}}$ of the choice of the surface Σ in the definition (2.6). Clearly, this is the consequence of the quantization condition $gg_m = 4\pi$ employed in the derivation of eq. (2.7).

A representation of the partition function (2.1), alternative to eq. (2.3) and more appropriate for the investigation of the Wilson loop, is the one in terms of the dynamical monopole densities [15]. It can be obtained by multiplying eq. (2.1) by the following unity:

$$\int \mathcal{D}\vec{\rho}\,\delta\left(\vec{\rho} - \vec{\rho}^{\mathcal{N}}\right) = \int \mathcal{D}\vec{\rho}\mathcal{D}\vec{\chi} \exp\left[ig_m \int d^3x \vec{\chi}\left(\vec{\rho} - \vec{\rho}^{\mathcal{N}}\right)\right],$$

so that the field $\vec{\chi}$ plays the role of the Lagrange multiplier. We obtain for the partition function:

$$\begin{aligned}\mathcal{Z}_{\rm mon} = &\int \mathcal{D}\vec{\rho}\mathcal{D}\vec{\chi} \exp\left[-\frac{g_m^2}{2}\int d^3x d^3y \vec{\rho}(\vec{x}) D_0(\vec{x}-\vec{y})\vec{\rho}(\vec{y}) + ig_m \int d^3x \vec{\chi}\vec{\rho}\right] \times \\ &\times \sum_{\mathcal{N}=0}^{\infty} \frac{\zeta^{\mathcal{N}}}{\mathcal{N}!}\left\langle \exp\left(-ig_m \int d^3x \vec{\chi}\vec{\rho}^{\mathcal{N}}\right)\right\rangle_{\rm mon},\end{aligned} \tag{2.8}$$

where the last sum is equal to

$$\exp\left[2\zeta \int d^3x \sum_{i=1}^{N(N-1)/2} \cos\left(g_m \vec{q}_i \vec{\chi}\right)\right]. \tag{2.9}$$

Accordingly, eq. (2.7) becomes replaced by

$$\begin{aligned}W_\alpha^{\mathcal{N}} \to W_\alpha = &\frac{1}{\mathcal{Z}_{\rm mon}} \int \mathcal{D}\vec{\rho}\mathcal{D}\vec{\chi} \exp\left\{-\frac{g_m^2}{2}\int d^3x d^3y \vec{\rho}(\vec{x}) D_0(\vec{x}-\vec{y})\vec{\rho}(\vec{y}) + \right. \\ &\left. + \int d^3x \left[ig_m \vec{\chi}\vec{\rho} + 2\zeta \sum_{i=1}^{N(N-1)/2} \cos\left(g_m \vec{q}_i \vec{\chi}\right) + i\vec{\mu}_\alpha \vec{\rho}\eta\right]\right\},\end{aligned}$$

and the full expression for the monopole contribution to the Wilson loop [instead of eq. (2.5)] reads $W(C)_{\rm mon} = \frac{1}{N}\sum_{\alpha=1}^{N} W_\alpha$.

Let us next introduce the magnetic field according to the formulae $\partial_\mu \vec{B}_\mu = \vec{\rho}$, $\varepsilon_{\mu\nu\lambda}\partial_\nu \vec{B}_\lambda = 0$. This yields

$$\begin{aligned}W_\alpha = &\frac{1}{\mathcal{Z}_{\rm mon}} \int \mathcal{D}\vec{B}_\mu \delta\left(\varepsilon_{\mu\nu\lambda}\partial_\nu \vec{B}_\lambda\right) \times \\ &\times \int \mathcal{D}\vec{\chi} \exp\left\{\int d^3x \left[-\frac{g_m^2}{2}\vec{B}_\mu^2 + ig_m \vec{\chi}\partial_\mu \vec{B}_\mu + 2\zeta \sum_{i=1}^{N(N-1)/2} \cos\left(g_m \vec{q}_i \vec{\chi}\right)\right] + \right. \\ &\left. + 4\pi i \vec{\mu}_\alpha \int_{\Sigma(C)} d\sigma_\mu \vec{B}_\mu\right\},\end{aligned} \tag{2.10}$$

where, as it was discussed, the surface $\Sigma(C)$ is arbitrary. The magnetic and the dual-photon fields can be integrated out of eq. (2.10) by solving the respective saddle-point equations. From now on in this section, we shall consider the contour C located in the (x, y)-plane. This naturally leads to the following Ansätze for the fields to be inserted into the saddle-point equations: $\vec{B}_\mu = \delta_{\mu 3}\vec{B}(z)$, $\vec{\chi} = \vec{\chi}(z)$. For the points (x, y) lying inside the contour C, these equations then read:

$$ {}_m\vec{\chi}' + g_m^2\vec{B} - 4\pi i\vec{\mu}_\alpha\delta(z) = 0\,, \tag{2.11} $$

$$ i\vec{B}' - 2\zeta\sum_{i=1}^{N(N-1)/2}\vec{q}_i\sin\left(g_m\vec{q}_i\vec{\chi}\right) = 0\,, \tag{2.12} $$

where $' \equiv d/dz$. (Note that differentiating eq. (2.11) and substituting the result into eq. (2.12), we obtain the equation

$$ \vec{\chi}'' - 2g_m\zeta\sum_{i=1}^{N(N-1)/2}\vec{q}_i\sin\left(g_m\vec{q}_i\vec{\chi}\right) = g\vec{\mu}_\alpha\delta'(z)\,, $$

which is the SU(N)-generalization of the respective saddle-point equation obtained in ref. [4].) These equations can be solved by adopting one extra Ansatz for the saddle-points of the fields, $\vec{B}(z) = \vec{\mu}_\alpha B(z)$, $\vec{\chi}(z) = \vec{\mu}_\alpha\chi(z)$, that due to the formula $\vec{\mu}_\alpha^2 = (N-1)/(2N)$ makes W_α α-independent. Let us next multiply eq. (2.12) by $\vec{\mu}_\alpha$, taking into account that for any α, $(N-1)$ positive roots yield the scalar product with $\vec{\mu}_\alpha$ equal to $1/2$, while the others are orthogonal to $\vec{\mu}_\alpha$. Equations (2.11), (2.12) then go over into

$$ 2i\psi' + g_m^2 B = 4\pi i\delta(z)\,, \qquad B' + 2i\zeta N\sin\psi = 0\,, \tag{2.13} $$

where $\psi \equiv g_m\chi/2$. One can straightforwardly check that the solution to this system of equations has the following form:

$$ B(z) = i\frac{8m_D}{g_m^2}\frac{\mathrm{e}^{-m_D|z|}}{1+\mathrm{e}^{-2m_D|z|}}\,, \qquad \psi(z) = 4\,\mathrm{sgn}\,z\cdot\arctan\left(\mathrm{e}^{-m_D|z|}\right). \tag{2.14} $$

Taking the value $B(0)$ for the evaluation of the Wilson loop, we obtain for the string tension:

$$ \sigma = 4\pi\cdot\frac{N-1}{2N}\frac{4m_D}{g_m^2} = \frac{8\pi}{g_m}\frac{N-1}{\sqrt{N}}\sqrt{\zeta}\,. \tag{2.15} $$

This result demonstrates explicitly the nonanalytic behavior of σ with g, but does not yield the correct overall numerical factor. That is because $B(z)$, necessary for this calculation, is defined ambiguously. The ambiguity originates from the exponentially large thickness of the string: $|z| < \mathcal{O}\left(m_D^{-1}\right)$. It is therefore unclear what value of $B(z)$ we should take: either $B(0)$, as it was done, or the one averaged over some range of z. It turns out that in the weak-field (low-density) limit, the problem can be solved for an arbitrarily-shaped surface, by using the representation in terms of the Kalb-Ramond field, which will be described in the next section.

3. SU(N) confining strings

One of the ways to introduce the Kalb-Ramond field is to do it first for the description of the dual photon, by considering this field as the field-strength tensor corresponding to the field $\vec{B}_\mu$, namely $\vec{B}_\mu = \frac{1}{2g_m}\varepsilon_{\mu\nu\lambda}\vec{h}_{\nu\lambda}$. The Wilson loop (2.10) then takes the form:[1]

$$W_\alpha = \frac{1}{\mathcal{Z}_{\rm mon}}\int \mathcal{D}\vec{h}_{\mu\nu}\delta\left(\partial_\mu \vec{h}_{\mu\nu}\right)\times$$
$$\times\int \mathcal{D}\vec{\chi}\exp\left\{\int d^3x\left[-\frac{1}{4}\vec{h}^2_{\mu\nu}+\frac{i}{2}\vec{\chi}\varepsilon_{\mu\nu\lambda}\partial_\mu\vec{h}_{\nu\lambda}+2\zeta\sum_{i=1}^{N(N-1)/2}\cos\left(g_m\vec{q}_i\vec{\chi}\right)\right]+\right.$$
$$\left.+\frac{ig}{2}\vec{\mu}_\alpha\int_{\Sigma(C)}d\sigma_{\mu\nu}\vec{h}_{\mu\nu}\right\}. \tag{3.1}$$

The field $\vec{\chi}$ can be further integrated out by solving the saddle-point equation of the form (2.12), where $\vec{B}'$ is replaced by $\frac{1}{2g_m}\varepsilon_{\mu\nu\lambda}\partial_\mu\vec{h}_{\nu\lambda}$ (and $\vec{\chi}$ depends on all three coordinates). Using the Ansatz $\vec{h}_{\mu\nu} = \vec{\mu}_\alpha h_{\mu\nu}$, we arrive at the following substitution in eq. (3.1):

$$\int d^3x\left[\frac{i}{2}\vec{\chi}\varepsilon_{\mu\nu\lambda}\partial_\mu\vec{h}_{\nu\lambda}+2\zeta\sum_{i=1}^{N(N-1)/2}\cos\left(g_m\vec{q}_i\vec{\chi}\right)\right]\Longrightarrow$$
$$\Longrightarrow -V\left[\vec{h}_{\mu\nu}\right]\equiv -2(N-1)\zeta\int d^3x\left[H_\alpha \mathrm{arcsinh}\, H_\alpha-\sqrt{1+H_\alpha^2}\right]. \tag{3.2}$$

Here, $H_\alpha \equiv \frac{1}{2g_m\zeta(N-1)}\vec{\mu}_\alpha\varepsilon_{\mu\nu\lambda}\partial_\mu\vec{h}_{\nu\lambda}$, and $V\left[\vec{h}_{\mu\nu}\right]$ is the multivalued potential of the Kalb-Ramond field (or of monopole densities). Similarly to the case of compact QED [6], it is the summation over branches of this potential which yields the summation over the world sheets $\Sigma(C)$ in eq. (3.1).

Another way to discuss the connection of the Kalb-Ramond field with the string world sheets is based on the semi-classical analysis of the saddle-point equations analogous to eqs. (2.13):

$$2i\partial_\mu\psi+\frac{g_m}{2}\varepsilon_{\mu\nu\lambda}h_{\nu\lambda} = 4\pi i\Sigma_\mu\,, \tag{3.3}$$
$$\frac{1}{2g_m}\varepsilon_{\mu\nu\lambda}\partial_\mu h_{\nu\lambda}+2i\zeta N\sin\psi = 0\,. \tag{3.4}$$

Here, $\Sigma_\mu \equiv \frac{1}{2}\varepsilon_{\mu\nu\lambda}\Sigma_{\nu\lambda}$, and the field ψ, by means of which the correspondence between the Kalb-Ramond field and the stationary surface is established, is defined by the relation $\vec{\chi} = \frac{2}{g_m}\vec{\mu}_\alpha\psi$. In this approach, based on the (auxiliary) field ψ, the summation over

[1]Note the following correspondence between the Coulomb interaction of monopole densities and the actions of the magnetic- and Kalb-Ramond fields

$$\frac{1}{4}\int d^3x\vec{h}^2_{\mu\nu} = \frac{g_m^2}{2}\int d^3x\vec{B}^2_\mu = \frac{g_m^2}{2}\int d^3xd^3y\vec{\rho}(\vec{x})D_0(\vec{x}-\vec{y})\vec{\rho}(\vec{y}).$$

branches of the potential $V\left[\vec{h}_{\mu\nu}\right]$ is equivalent to the following procedure. One should restrict oneself to the domain $|\psi| \le \pi$ and solve eqs. (3.3), (3.4) with the conditions

$$\lim_{|\vec{x}|\to\infty} \psi(\vec{x}) = 0\,, \tag{3.5}$$

$$\lim_{\varepsilon\to 0} \left\{\psi\left[\vec{x}(\xi)+\varepsilon\vec{n}(\xi)\right] - \psi\left[\vec{x}(\xi)-\varepsilon\vec{n}(\xi)\right]\right\} = 2\pi\,, \tag{3.6}$$

where $\vec{n}(\xi)$ is the normal vector to Σ at an arbitrary point $\vec{x}(\xi)$. After that, one should get rid of the so-appearing Σ-dependence of W_α by extremizing the latter with respect to $\vec{x}(\xi)$. Owing to the formula

$$\delta \int d\sigma_{\mu\nu} h_{\mu\nu}\left[\vec{x}(\xi)\right] = \int d\sigma_{\mu\nu} H_{\mu\nu\lambda}\left[\vec{x}(\xi)\right]\delta x_\lambda(\xi)\,,$$

such an extremization is equivalent to the definition of the extremal surface through the equation $H_{\mu\nu\lambda}\left[\vec{x}(\xi)\right] = 0$. By virtue of eqs. (3.4), (3.6), $H_{\mu\nu\lambda}$ indeed vanishes on both sides of the stationary surface. The reason for that is the following distinguished property of the extremal surface: ψ is equal to π on one of its sides and to $-\pi$ on the other, whereas for any other surface, $|\psi|$ is smaller than π on one side and larger than π on the other side. Therefore, according to eq. (3.4), at both sides of the extremal surface, $\varepsilon_{\mu\nu\lambda}\partial_\mu h_{\nu\lambda} = 0$. This is equivalent to the equation $\varepsilon_{\mu\nu\lambda}H_{\mu\nu\lambda} = 0$ and hence (upon the multiplication of both sides by $\varepsilon_{\mu'\nu'\lambda'}$) to $H_{\mu\nu\lambda} = 0$. [In particular, for a flat contour the extremal surface is the flat surface inside this contour. This can explicitly be seen from the solutions (2.14) of eqs. (2.13). Namely, at any z such that $|z| \ll m_D^{-1}$, $\psi(z) \simeq \pi\cdot\mathrm{sgn}z$, $B'(z)\simeq 0$, where both equalities hold with the exponential accuracy. The equation $B'(z)\simeq 0$ is equivalent to the condition of vanishing of $H_{\mu\nu\lambda}$ at both sides of the flat surface.]

According to the Hodge decomposition theorem, the general form of $\vec{h}_{\mu\nu}$ is $\vec{h}_{\mu\nu} = \partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu + \varepsilon_{\mu\nu\lambda}\partial_\lambda\vec{\phi}$. The constraint $\partial_\mu\vec{h}_{\mu\nu} = 0$, imposed in eq. (3.1), is equivalent to setting $\vec{A}_\mu$ equal zero. From now on, we shall promote $\vec{h}_{\mu\nu}$ to include also the fields of free photons, $\vec{A}_\mu$, by abolishing this constraint. Let us further go into the weak-field limit, $|H_\alpha| \ll 1$. Using the Cauchy inequality, we have

$$|H_\alpha| \le \frac{|\vec{\mu}_\alpha|\,|\vec{\rho}|}{\zeta(N-1)} = \frac{|\vec{\rho}|}{\zeta\sqrt{2N(N-1)}}\,,$$

so that the weak-field limit is equivalent to the following low-density approximation: $|\vec{\rho}| \ll \zeta\sqrt{2N(N-1)}$. To understand in what sense this inequality implies the low-density approximation, note that the mean monopole density stemming from eq. (2.3) reads:

$$|\vec{\rho}|_{\mathrm{mean}} = \frac{\partial \ln \mathcal{Z}_{\mathrm{mon}}}{\mathcal{V}\partial\ln\zeta} \simeq \zeta N(N-1)\,,$$

where $\mathcal{V}$ is the three-volume occupied by the system. Therefore, at large-N, the low-density approximation implies that $|\vec{\rho}|$ should be of the order N times smaller than its mean value.

In the weak-field limit, we now have the following expression for the total Wilson loop, where the constraint $\partial_\mu\vec{h}_{\mu\nu} = 0$ is removed:

$$W\left(C,\Sigma\right)^{\mathrm{tot}}_{\mathrm{weak-field}} = \frac{1}{\mathcal{Z}^{\mathrm{tot}}}\int \mathcal{D}\vec{h}_{\mu\nu}\exp\left\{-\int d^3x\left[\frac{1}{12m_D^2}\vec{H}^2_{\mu\nu\lambda} + \frac{1}{4}\vec{h}^2_{\mu\nu} - \frac{ig}{2}\vec{\mu}_\alpha\vec{h}_{\mu\nu}\Sigma_{\mu\nu}\right]\right\}. \tag{3.7}$$

Here, $\mathcal{Z}^{\rm tot}$ is given by the same integral over $\vec{h}_{\mu\nu}$-field, but with $\Sigma_{\mu\nu}$ set to zero, and $\vec{H}_{\mu\nu\lambda} = \partial_\mu \vec{h}_{\nu\lambda} + \partial_\lambda \vec{h}_{\mu\nu} + \partial_\nu \vec{h}_{\lambda\mu}$ is the Kalb-Ramond field-strength tensor. (The apparent Σ-dependence of the r.h.s. of eq. (3.7) is due to the fact that in course of the $\vec{h}_{\mu\nu}$-expansion of $V\left[\vec{h}_{\mu\nu}\right]$, only one branch of this potential has been taken into account. As it was discussed above, the Σ-dependence disappears upon the summation over all the branches.) Owing to the Hodge decomposition theorem for $\vec{h}_{\mu\nu}$, eq. (3.7) is obviously factorized as $W(C,\Sigma)^{\rm tot}_{\rm weak-field} = W_\alpha W_\alpha^{\rm phot}$, where the free photonic contribution to the Wilson loop reads

$$W_\alpha^{\rm phot} = \exp\left[-g^2\frac{N-1}{4N}\oint_C dx_\mu \oint_C dx'_\mu D_0\left(\vec{x}-\vec{x}'\right)\right]. \tag{3.8}$$

Doing the integration over $\vec{h}_{\mu\nu}$ in eq. (3.7), we obtain

$$W(C,\Sigma)^{\rm tot}_{\rm weak-field} = \exp\Biggl\{-g^2\frac{N-1}{4N}\Biggl[\oint_C dx_\mu \oint_C dx'_\mu D_{m_D}(\vec{x}-\vec{x}') + \frac{m_D^2}{2}\int_{\Sigma(C)} d\sigma_{\mu\nu}(\vec{x}(\xi)) \times$$
$$\times \int_{\Sigma(C)} d\sigma_{\mu\nu}(\vec{x}(\xi'))D_{m_D}\left(\vec{x}(\xi)-\vec{x}(\xi')\right)\Biggr]\Biggr\}, \tag{3.9}$$

where $D_{m_D}(\vec{x}) = {\rm e}^{-m_D|\vec{x}|}/(4\pi|\vec{x}|)$ is the $3D$ Yukawa propagator. Note that the free photonic contribution is completely canceled out of this expression, i.e., it is only the dual photon (of the mass m_D) which mediates the $C\times C$- and $\Sigma\times\Sigma$-interactions.

One can further expand the nonlocal string effective action

$$S_{\rm str} = (gm_D)^2\frac{N-1}{8N}\int_{\Sigma(C)} d\sigma_{\mu\nu}(\vec{x}(\xi))\int_{\Sigma(C)} d\sigma_{\mu\nu}(\vec{x}(\xi'))D_{m_D}\left(\vec{x}(\xi)-\vec{x}(\xi')\right) \tag{3.10}$$

in powers of the derivatives with respect to the world-sheet coordinates ξ^a's. Note that the actual parameter of this expansion is $1/(m_D R)^2$, where $R\sim\sqrt{{\rm Area}(\Sigma)}$ is the size of Σ (see the discussion below). The resulting quasi-local action reads (cf. refs. [19, 7]):

$$S_{\rm str} = \sigma\int d^2\xi\sqrt{\mathsf{g}} + \alpha^{-1}\int d^2\xi\sqrt{\mathsf{g}}\mathsf{g}^{ab}(\partial_a t_{\mu\nu})(\partial_b t_{\mu\nu}) + \kappa\int d^2\xi\sqrt{\mathsf{g}}\mathcal{R} + \mathcal{O}\left(\frac{\sigma}{m_D^4 R^2}\right). \tag{3.11}$$

Here, $\partial_a \equiv \partial/\partial\xi^a$, and the following quantities characterize Σ: $\mathsf{g}_{ab}(\xi) = (\partial_a x_\mu(\xi))(\partial_b x_\mu(\xi))$ is the induced-metric tensor, $\mathsf{g} = \det\|\mathsf{g}^{ab}\|$, $t_{\mu\nu}(\xi) = \varepsilon^{ab}(\partial_a x_\mu(\xi))(\partial_b x_\nu(\xi))/\sqrt{\mathsf{g}}$ is the extrinsic-curvature tensor, $\mathcal{R} = \left(\partial^a\partial_a \ln\sqrt{\mathsf{g}}\right)/\sqrt{\mathsf{g}}$ is the (conformal-gauge expression for the) scalar (or intrinsic) curvature.[2] The string coupling constants here read $\sigma = 2\pi^2\frac{N-1}{\sqrt{N}}\frac{\sqrt{\zeta}}{g_m}$, $\alpha^{-1} = -3\kappa/2 = -\frac{\pi^2(N-1)}{4g_m^3 N^{3/2}\sqrt{\zeta}}$. In the weak-field (or low-density) approximation, we can therefore fix the proportionality factor $2\pi^2$ in σ, that is close to the factor 8π of eq. (2.15) which, however, could have not been fixed within the method of section 2. Another important fact is that (in the weak-field approximation) the string tension and higher string coupling constants are the same for all large enough surfaces $\Sigma(C)$. [The

[2]The second term on the r.h.s. of eq. (3.11) describes the so-called rigidity of the string (see e.g. ref. [5]), while the third term is known to be a full derivative and therefore it does not actually contribute to the string effective action.

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words "large enough" here mean the validity of the inequality $m_D R \gg 1$, i.e., the string length R should significantly exceed the exponentially large string thickness m_D^{-1}. Indeed, the general n-th term of the derivative (or curvature) expansion of $S_{\rm str}$ is of the order of $\sigma m_D^2 R^4 (m_D R)^{-2n}$.] It is, however, worth noting that some of the terms in the expansion (3.11) vanish at the surface of the minimal area corresponding to a given contour C (e.g., at the flat surface, in the case considered in section 2). This concerns, for instance, the rigidity term.[3]

Let us finally demonstrate how to introduce the Kalb-Ramond field in the way alternative to its definition via the $\vec{B}_\mu$-field and incorporating automatically the free-photonic contribution to the Wilson loop. It is based on the direct application of eqs. (2.1) and (2.7), that yields

$$W_\alpha = \sum_{\mathcal{N}=0}^{\infty} \frac{\zeta^{\mathcal{N}}}{\mathcal{N}!} \left\langle \exp\left[-\frac{g_m^2}{2} \int d^3x d^3y \vec{\rho}^{\mathcal{N}}(\vec{x}) D_0(\vec{x}-\vec{y}) \vec{\rho}^{\mathcal{N}}(\vec{y}) + i\vec{\mu}_\alpha \int d^3x \vec{\rho}^{\mathcal{N}} \eta\right]\right\rangle_{\rm mon}$$
$$= \frac{1}{\mathcal{Z}_{\rm mon}} \int \mathcal{D}\vec{\varphi} \exp\left\{-\int d^3x \left[\frac{1}{2}\left(\partial_\mu \vec{\varphi} - \frac{\vec{\mu}_\alpha}{g_m}\partial_\mu \eta\right)^2 - 2\zeta \sum_{i=1}^{N(N-1)/2} \cos\left(g_m \vec{q}_i \vec{\varphi}\right)\right]\right\}, \quad (3.12)$$

where $\vec{\varphi} = \vec{\chi} + \frac{\vec{\mu}_\alpha}{g_m}\eta$. One can further use the following formula which shows explicitly how the Kalb-Ramond field unifies the monopole and the free-photonic contributions to the Wilson loop:

$$\exp\left[-\frac{1}{2}\int d^3x \left(\partial_\mu \vec{\varphi} - \frac{\vec{\mu}_\alpha}{g_m}\partial_\mu \eta\right)^2 - g^2 \frac{N-1}{4N} \oint_C dx_\mu \oint_C dx'_\mu D_0(\vec{x}-\vec{x}')\right] =$$
$$= \int \mathcal{D}\vec{h}_{\mu\nu} \exp\left[-\int d^3x \left(\frac{1}{4}\vec{h}_{\mu\nu}^2 + \frac{i}{2}\vec{\varphi}\varepsilon_{\mu\nu\lambda}\partial_\mu \vec{h}_{\nu\lambda} - \frac{ig}{2}\vec{\mu}_\alpha \vec{h}_{\mu\nu}\Sigma_{\mu\nu}\right)\right]. \quad (3.13)$$

In fact, one can prove that both sides of this formula are equal to[4]

$$\exp\left[-\frac{1}{2}\int d^3x \left(\partial_\mu \vec{\varphi} + g\vec{\mu}_\alpha \Sigma_\mu\right)^2\right].$$

Inserting further eq. (3.13) into eq. (3.12), we obtain

$$W_\alpha = \frac{1}{\mathcal{Z}^{\rm tot}} \int \mathcal{D}\vec{h}_{\mu\nu}\mathcal{D}\vec{\varphi} \exp\left[-\int d^3x \left(\frac{1}{4}\vec{h}_{\mu\nu}^2 + \frac{i}{2}\vec{\varphi}\varepsilon_{\mu\nu\lambda}\partial_\mu \vec{h}_{\nu\lambda} - 2\zeta \sum_{i=1}^{N(N-1)/2} \cos\left(g_m \vec{q}_i \vec{\varphi}\right) - \right.\right.$$
$$\left.\left. - \frac{ig}{2}\vec{\mu}_\alpha \vec{h}_{\mu\nu}\Sigma_{\mu\nu}\right)\right].$$

Denoting $\vec{\chi} \equiv -\vec{\varphi}$, we arrive back to eq. (3.1) with the constraint $\partial_\mu \vec{h}_{\mu\nu} = 0$ abolished, as it should be.

[3] Indeed, one can prove the equality $\mathrm{g}^{ab}(\partial_a t_{\mu\nu})(\partial_b t_{\mu\nu}) = (D^a D_a x_\mu)(D^b D_b x_\mu)$, where $D_a D_b x_\mu = \partial_a \partial_b x_\mu - \Gamma^c_{ab}\partial_c x_\mu$, and Γ^c_{ab} is the Christoffel symbol corresponding to the metric g^{ab}. On the other hand, the surface of the minimal area is defined by the equation $D^a D_a x_\mu(\xi) = 0$ together with the respective boundary condition at the contour C.

[4] Obviously, in the noncompact case, when monopoles are disregarded and $\vec{h}_{\mu\nu} = \partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu$, the r.h.s. of eq. (3.13) yields the free-photonic contribution to the Wilson loop, eq. (3.8).

4. Summary

The present paper has been devoted to the theory of confining strings which describes the string representation of the SU(N)-analogue of $3D$ compact QED. We have first found the string tension of the flat Wilson loop, that has been done by the evaluation of the mean magnetic field inside the contour of the loop, using the saddle-point analysis. The obtained result contains an ambiguity in the overall constant factor, that is due to the large thickness of the confining string which makes the definition of the mean magnetic field ambiguous. We have then developed the SU(N)-theory of confining strings which, in its weak-field limit (corresponding to the limit of low monopole densities), enables one to fix the above-mentioned factor. Moreover, this theory enables one also to find the coupling constants of the terms in the expansion of the nonlocal string effective action which are higher in the derivatives than the Nambu-Goto term. The theory of confining strings has been introduced in two various ways. One of these is to consider the Kalb-Ramond field as the field-strength tensor corresponding to the magnetic field produced by monopoles. The free photons can further be included by abolishing a certain constraint imposed on the Kalb-Ramond field. In the other approach, the Kalb-Ramond field appears directly as the field unifying the monopole and free-photonic contributions to the Wilson loop, using the definition of the Wilson loop in the monopole plasma. Note finally that the above-mentioned results can be readily generalized to the case of $3D$ Georgi-Glashow model with not-infinitely-heavy Higgs field, along with the lines of ref. [15]. It also looks reasonable to address interactions of Wilson loops and to compare the so-obtained results with those which were recently obtained in QCD within the Stochastic Vacuum Model [20].

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